Math 4200
Monday November 2
3.3 Laurent series. We'll prove the Laurent series theorem for analytic functions in annuli, which we illustrated on Friday with rational function examples.

Announcements:

Laurent Series Theorem For $0 \le R_1 < R_2$ let

$$A = \{ z \in \mathbb{C} \mid R_1 < |z - z_0| < R_2 \}$$

be an open annulus (or punctured disk in case $R_1 = 0$). Then (1) and (2) below are equivalent, and the uniqueness of Laurent coefficients (3) also holds:

(1) $f : A \to \mathbb{C}$ is analytic.

(2) $f(z)$ has a power series expansion using non-negative and negative powers of $(z - z_0)$:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{m=1}^{\infty} \frac{a_{-m}}{(z - z_0)^m}.$$

$$:= S_1(z) + S_2(z).$$

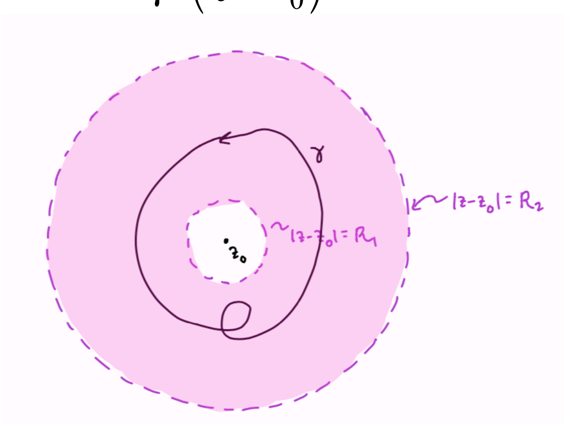
Here $S_1(z)$ converges for $|z - z_0| < R_2$ and uniformly absolutely for $|z - z_0| \le r_2 < R_2$. And $S_2(z)$ converges for $|z - z_0| > R_1$, and uniformly for $|z - z_0| \ge r_1 > R_1$.

(3) The Laurent coefficients a_k, $k \in \mathbb{Z}$ are uniquely determined by f. Specifically, if γ is any p.w. C^1 contour in A, with $I(\gamma, z_0) = 1$, e.g. any circle of radius r, with $R_1 < r < R_2$, then

$$a_{-1} = \frac{1}{2 \pi i} \int_{\gamma} f(\zeta) d\zeta$$

and more generally, each a_k is given by

$$a_k = \frac{1}{2 \pi i} \int_{\gamma} \frac{f(\zeta)}{(\zeta - z_0)^{k+1}} d\zeta$$



We'll discuss $(2) \Rightarrow (1)$, $(2) \Rightarrow (3)$ on this page, and $(1) \Rightarrow (2)$ on the next page.

proof of (1) ⇒ (2) in the Laurent series theorem:

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$$:= S_1(z) + S_2(z).$$

Here $S_1(z)$ converges uniformly absolutely for any compact subdisk $|z - z_0| \le r_2 < R_2$.

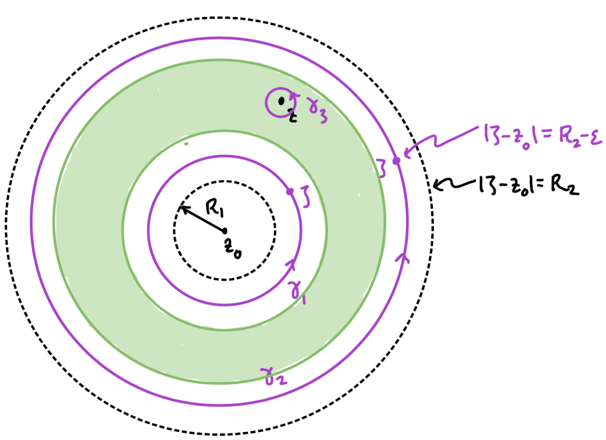
And $S_2(z)$ converges uniformly absolutely for $|z - z_0| \ge r_1 > R_1$.

proof: Let $\varepsilon > 0$. Consider z in the compact subannulus $R_1 + 2\varepsilon \le |z - z_0| \le R_2 - 2\varepsilon$. Let γ_1 be the circle $|\zeta - z_0| = R_1 + \varepsilon$, let γ_2 be the circle $|\zeta - z_0| = R_2 - \varepsilon$, and let γ_3 be a concentric circle arround z or radius less than ε. (See figure.) Use the section 2.2 replacement theorem for the first equation below, and then the CIF for the second one:

$$\int_{\gamma_2} \frac{f(\zeta)}{\zeta - z} d\zeta = \int_{\gamma_1} \frac{f(\zeta)}{\zeta - z} d\zeta + \int_{\gamma_3} \frac{f(\zeta)}{\zeta - z} d\zeta \quad ,$$

$$f(z) = \frac{1}{2\pi i} \int_{\gamma_2} \frac{f(\zeta)}{\zeta - z} d\zeta - \frac{1}{2\pi i} \int_{\gamma_1} \frac{f(\zeta)}{\zeta - z} d\zeta$$

Now use our geometric series wizardry to find the Laurent expansion for $f(z)$!



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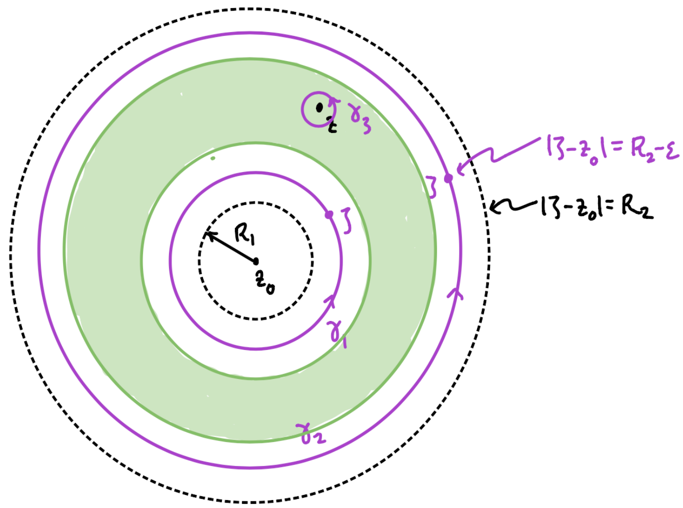


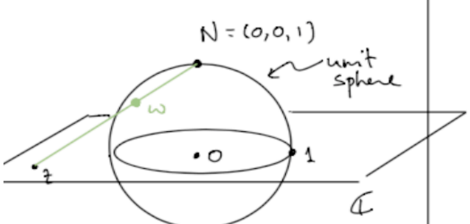
On the next two pages we use Laurent series to classify *isolated singularities,* and give equivalent geometric conditions which characterize the three kinds: removable singularities, poles, essential singularities. And we will revert to the text's lettering for the coefficients of the positive and negative powers in a Laurent series. $f(z)$ has an *isolated singularity at* z_0 means that there is some radius $r > 0$ so that f is analytic in the punctured disk $D(z_0, r) \setminus \{z_0\}$ We write the Laurent series as

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad + \sum_{m=1}^{\infty} \frac{b_m}{(z - z_0)^m}$$

Isolated singularities table

Let f analytic in $D(z_0;r)\setminus\{z_0\}$, some $r>0$

type of singularity @ z_0	Laurent series definition	geometric characterization (behavior of f near z_0)
removable (because f extends to be analytic @ z_0)	$f(z)=\sum_{n=0}^{\infty} a_n(z-z_0)^n$ (no negative powers in L.S.)	any of: ① $\lim_{z\to z_0} f(z)=L\in\mathbb{C}$ exists ② $\lvert f(z)\rvert\le M$ $\forall\ 0<\lvert z-z_0\rvert\le\rho$ some $0<\rho<r$ ③ $\lim_{z\to z_0} f(z)(z-z_0)=0$
pole (North pole!) of order $N\in\mathbb{N}$; simple pole if $N=1$	$f(z)=\sum_{n=0}^{\infty} a_n(z-z_0)^n+\sum_{m=1}^{N}\frac{b_m}{(z-z_0)^m}$ with $b_N\neq 0$	① $\lim_{z\to z_0} f(z)=\infty$ (the north pole of the Riemann sphere) ② $\exists\ N\in\mathbb{N}$ s.t. $g(z)=(z-z_0)^N f(z)$ has a removable singularity at $z=z_0$, with $g(z_0)\neq 0$



$z=x+iy$

$w=\left(\frac{2x}{1+x^2+y^2},\frac{2y}{1+x^2+y^2},\frac{-1+x^2+y^2}{1+x^2+y^2}\right)$

$x+iy=\frac{w_1}{1-w_3}+i\frac{w_2}{1-w_3}$ $(w_3\neq 0)$

$\lvert z\rvert\to\infty$ iff $w\to(0,0,1)$

type of singularity @ z_0	Laurent series def.	geometric characterization
essential singularity	$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{m=1}^{\infty} \frac{b_m}{(z-z_0)^m}$ with $b_{m_j} \neq 0$ for some seq. $\{m_j\} \to \infty$.	$\forall\, 0<\rho<r$, $\overline{f(D(z_0;\rho) \setminus \{z_0\})} = \mathbb{C}$! (In fact, more is true and is called "Picard's Thm": $f(D(z_0;\rho) \setminus \{z_0\})$ contains all of $\mathbb{C}$ except for at most a single point ! $\forall\, 0<\rho<r$) e.g. $f(z) = e^{\frac{1}{z}}$ @ $z_0 = 0$: $f(D(0;\rho) \setminus \{0\}) = \mathbb{C} \setminus \{0\}$ $\forall \rho > 0$.